

THREE-PARAMETER COMPLEX HADAMARD MATRICES OF ORDER 6.

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ABSTRACT. A three-parameter family of complex Hadamard matrices of order 6 is presented. It significantly extends the set of closed form complex Hadamard matrices of this order, and in particular contains all previously described one- and two-parameter families as subfamilies.

1. INTRODUCTION

Complex Hadamard matrices have turned out hard to classify, with current classifications being incomplete for order 6 and higher. For order 6, there is evidence for a four-parameter family [1, 2], but up till now only zero-, one- and two-parameter subfamilies have been obtained on closed form, as reviewed in [3, 4]. Recent progress includes the construction of three two-parameter, nonaffine families [5, 6] that contain the one-parameter families as subfamilies, and has resulted in an overall picture of five, partially overlapping, two-parameter families of complex Hadamard matrices of this order.

A further step towards a more comprehensive classification was taken in [7], where it was shown that any complex Hadamard matrix of order 6 is equivalent to (or equals) a Hadamard matrix for which either all (the H_2 -reducible case) or none of its nine 2×2 submatrices are Hadamard. In the present paper, a complete characterization of the H_2 -reducible Hadamard matrices is given. The result is a three-parameter family which has all the previously known (one- and) two-parameter families as subfamilies.

2. PRELIMINARIES

An $N \times N$ matrix H with complex elements h_{ij} is Hadamard if all elements have modulus one, $|h_{ij}| = 1$, and if

$$(2.1) \quad HH^\dagger = H^\dagger H = NE$$

(the unitarity constraint), where E is the unit matrix in N dimensions. Two Hadamard matrices are termed equivalent, $H_1 \sim H_2$, if they can

be related through

$$(2.2) \quad H_2 = D_2 P_2 H_1 P_1 D_1$$

where D_1 and D_2 are diagonal unitary matrices, and P_1 and P_2 are permutation matrices. A set of equivalent Hadamard matrices can be represented by a dephased matrix, with ones in the first row and the first column.

The present paper will be concerned with Hadamard matrices which are reducible in the following sense.

Definition 1. A complex Hadamard matrix of order 6 is H_2 -reducible if it is equivalent to a Hadamard matrix for which all the nine 2×2 submatrices are Hadamard.

H_2 -reducible Hadamard matrices are more prevalent than might be thought. The quite general nature of these matrices is illustrated by the following theorem that was proven in [7].

Theorem 2. *Let H be a complex Hadamard matrix of order 6, with elements h_{ij} , $i, j = 1, \dots, 6$. If there exists an order 2 submatrix $\begin{pmatrix} h_{ij} & h_{ik} \\ h_{lj} & h_{lk} \end{pmatrix}$ that is Hadamard, then H is H_2 -reducible.*

Since the submatrix referred to in Theorem 2 has the (unique) dephased form

$$(2.3) \quad F_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

H_2 -reducible Hadamard matrices are easily identified:

Corollary 3. *Let H be a complex Hadamard matrix of order 6. H is H_2 -reducible if, and only if, its dephased form has at least one element equal to -1.*

It follows from the corollary that all the currently known one- and two-parameter families in six dimensions ($F_6^{(2)}$, $(F_6^{(2)})^T$, $D_6^{(1)}$ [8], $B_6^{(1)}$ [9], $M_6^{(1)}$ [10], $X_6^{(2)}$, $(X_6^{(2)})^T$ [5] and $K_6^{(2)}$ [6], in the notation of [3, 4]) are families of H_2 -reducible Hadamard matrices. On the other hand, the single, isolated matrix $S_6^{(0)}$ is not H_2 -reducible.

A general, H_2 -reducible Hadamard is equivalent to a Hadamard matrix on the dephased form (see [7])

$$(2.4) \quad H = \begin{pmatrix} F_2 & Z_1 & Z_2 \\ Z_3 & a & b \\ Z_4 & c & d \end{pmatrix}$$

where each of the (Hadamard) matrices Z_i is fully determined by a single complex number z_i of modulus one, $|z_i| = 1$,

$$(2.5) \quad \begin{aligned} Z_1 &= \begin{pmatrix} 1 & 1 \\ z_1 & -z_1 \end{pmatrix} & Z_2 &= \begin{pmatrix} 1 & 1 \\ z_2 & -z_2 \end{pmatrix} \\ Z_3 &= \begin{pmatrix} 1 & z_3 \\ 1 & -z_3 \end{pmatrix} & Z_4 &= \begin{pmatrix} 1 & z_4 \\ 1 & -z_4 \end{pmatrix} \end{aligned},$$

and where a, b, c and d are Hadamard matrices of order 2. Not all matrices of the general form (2.4) will be Hadamard, and the additional conditions on the matrix elements will now be investigated.

3. THE UNITARITY CONSTRAINTS

In order to develop an exhaustive parametrization of the H_2 -reducible Hadamard matrices on the standard form (2.4), the unitarity constraints on H and its submatrices are first explored. In a second step, the additional constraints imposed by the unimodularity of the elements of H are investigated.

Let e be the unit matrix in two dimensions.

Proposition 4. *Let H be an H_2 -reducible Hadamard matrix on the standard form (2.4). Then*

$$H = \begin{pmatrix} F_2 & Z_1 & Z_2 \\ Z_3 & \frac{1}{2}Z_3AZ_1 & \frac{1}{2}Z_3BZ_2 \\ Z_4 & \frac{1}{2}Z_4BZ_1 & \frac{1}{2}Z_4AZ_2 \end{pmatrix}$$

where

$$\begin{aligned} A &= F_2\left(-\frac{1}{2}e + i\frac{\sqrt{3}}{2}\Lambda\right) \\ B &= F_2\left(-\frac{1}{2}e - i\frac{\sqrt{3}}{2}\Lambda\right) \end{aligned}$$

and where the 2×2 matrix Λ is unitary, $\Lambda^\dagger \Lambda = \Lambda \Lambda^\dagger = e$, and self-adjoint, $\Lambda^\dagger = \Lambda$.

Proof. In (2.4), let $a = \frac{1}{2}Z_3AZ_1$, $b = \frac{1}{2}Z_3BZ_2$, $c = \frac{1}{2}Z_4AZ_1$ and $d = \frac{1}{2}Z_4BZ_2$. In terms of A, B, C and D , the full set of unitarity constraints

on H take the form

$$(3.1) \quad \begin{cases} A + B = -F_2 \\ C + D = -F_2 \\ A + C = -F_2 \\ B + D = -F_2 \end{cases}$$

and

$$(3.2) \quad \begin{cases} AA^\dagger + BB^\dagger = 4e \\ CC^\dagger + DD^\dagger = 4e \\ AC^\dagger + BD^\dagger = -2e \end{cases} \quad \begin{cases} A^\dagger A + C^\dagger C = 4e \\ B^\dagger B + D^\dagger D = 4e \\ A^\dagger B + C^\dagger D = -2e \end{cases}$$

Note that these conditions are independent of z_1, z_2, z_3 and z_4 . It follows from (3.1) that $D = A$ and $C = B$. The relations (3.2) can therefore be reduced to

$$(3.3) \quad \begin{cases} (A+B)(A+B)^\dagger = 2e \\ (A+B)^\dagger(A+B) = 2e \\ (A-B)(A-B)^\dagger = 6e \\ (A-B)^\dagger(A-B) = 6e \end{cases}$$

In view of the constraint $A+B = -F_2$ (from (3.1)), the first two of these relations are always satisfied. In terms of $\Lambda \equiv -iF_2(A-B)/(2\sqrt{3})$, the last two relations imply that Λ is unitary, $\Lambda^\dagger\Lambda = \Lambda\Lambda^\dagger = e$. Finally, by assumption, $a^\dagger a = b^\dagger b = 2e$ so that $A^\dagger A = B^\dagger B = 2e$. As a result, $(A+B)^\dagger(A-B) + (A-B)^\dagger(A+B) = 0$ or, in terms of Λ , $\Lambda - \Lambda^\dagger = 0$. Solving for A and B in terms of F_2 and Λ completes the proof. \square

Lemma 5. *If a 2×2 matrix Λ is unitary and self-adjoint, either $\Lambda = \pm e$ or*

$$\Lambda = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \bar{\Lambda}_{12} & -\Lambda_{11} \end{pmatrix} = \begin{pmatrix} \cos \theta & e^{i\phi} \sin \theta \\ e^{-i\phi} \sin \theta & -\cos \theta \end{pmatrix}$$

with $\theta \in [0, 2\pi)$ and $\phi \in [0, 2\pi)$.

Proof. Since Λ is self-adjoint, its diagonal elements Λ_{11} and Λ_{22} are real, and $\Lambda_{21} = \bar{\Lambda}_{12}$. Furthermore, since Λ is unitary,

$$\Lambda^\dagger \Lambda = \begin{pmatrix} \Lambda_{11}^2 + |\Lambda_{12}|^2 & \Lambda_{12}(\Lambda_{11} + \Lambda_{22}) \\ \bar{\Lambda}_{12}(\Lambda_{11} + \Lambda_{22}) & \Lambda_{22}^2 + |\Lambda_{12}|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The off-diagonal elements vanish if either $\Lambda_{22} = -\Lambda_{11}$ or $\Lambda_{12} = \Lambda_{21} = 0$. In the first case Λ is traceless, with $\Lambda_{11}^2 + |\Lambda_{12}|^2 = 1$, i.e. Λ can be parametrized as

$$\Lambda = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \bar{\Lambda}_{12} & -\Lambda_{11} \end{pmatrix} = \begin{pmatrix} \cos \theta & e^{i\phi} \sin \theta \\ e^{-i\phi} \sin \theta & -\cos \theta \end{pmatrix}$$

with $\theta \in [0, 2\pi)$ and $\phi \in [0, 2\pi)$, and $\det \Lambda = -1$. In the second case Λ is diagonal with $\Lambda_{11}^2 = \Lambda_{22}^2 = 1$. The possibility that $\Lambda_{11} = -\Lambda_{22}$ is already included in the first case, leaving $\Lambda = \pm e$ as the only new possibilities, and for which $\det \Lambda = 1$. \square

Remark. In more general terms, if a 2×2 unitary matrix Λ is selfadjoint, either $\Lambda \in SU(2)$, or $i\Lambda \in SU(2)$. In particular, the parametrization for Λ given in Lemma 5 is directly related to the standard parametrization of $SU(2)$ matrices.

Corollary 6. *The matrices A and B of Proposition 4 either have the form (for $\Lambda = e$)*

$$(3.4) \quad A = \omega F_2 \quad \text{and} \quad B = \omega^2 F_2$$

or (for $\Lambda = -e$)

$$(3.5) \quad A = \omega^2 F_2 \quad \text{and} \quad B = \omega F_2$$

with $\omega = -1/2 + i\sqrt{3}/2 = \exp(2\pi i/3)$, or otherwise (for $\Lambda \neq \pm e$)

$$(3.6) \quad A = \begin{pmatrix} A_{11} & A_{12} \\ \bar{A}_{12} & -\bar{A}_{11} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_{11} & B_{12} \\ \bar{B}_{12} & -\bar{B}_{11} \end{pmatrix}$$

where

$$\begin{aligned} A_{11} &= -\frac{1}{2} + i\frac{\sqrt{3}}{2}(\cos \theta + e^{-i\phi} \sin \theta) \\ A_{12} &= -\frac{1}{2} + i\frac{\sqrt{3}}{2}(-\cos \theta + e^{i\phi} \sin \theta) \end{aligned}$$

and $B = -F_2 - A$.

At this point, all unitarity constraints on the matrix H and its submatrices have been accounted for. Note that although the matrices A and B satisfy the unitarity constraints, they will in general not be Hadamard (the modulus of the matrix elements will not be equal to one).

4. THE UNIMODULARITY CONSTRAINTS

The additional condition that all elements of H should be of unit modulus can now be imposed.

Proposition 7. *Let H be an H_2 -reducible Hadamard matrix on the form (2.4), and let A and B be as in Proposition 4 and Corollary 6.*

For A and B according to (3.4) or (3.5), the elements of a , b , c and d are of unit modulus if

$$\begin{aligned}
 (1 - z_1^2)(1 - z_3^2) &= 0 \\
 (1 - z_2^2)(1 - z_4^2) &= 0 \\
 (1 - z_1^2)(1 - z_4^2) &= 0 \\
 (4.1) \quad (1 - z_2^2)(1 - z_3^2) &= 0.
 \end{aligned}$$

For A and B according to (3.6), the elements of a , b , c and d are of unit modulus if

$$\begin{aligned}
 -A_{11}^2 + z_1^2 A_{12}^2 + z_3^2 \bar{A}_{12}^2 - z_1^2 z_3^2 \bar{A}_{11}^2 &= 0 \\
 -B_{11}^2 + z_2^2 B_{12}^2 + z_3^2 \bar{B}_{12}^2 - z_2^2 z_3^2 \bar{B}_{11}^2 &= 0 \\
 -B_{11}^2 + z_1^2 B_{12}^2 + z_4^2 \bar{B}_{12}^2 - z_1^2 z_4^2 \bar{B}_{11}^2 &= 0 \\
 (4.2) \quad -A_{11}^2 + z_2^2 A_{12}^2 + z_4^2 \bar{A}_{12}^2 - z_2^2 z_4^2 \bar{A}_{11}^2 &= 0.
 \end{aligned}$$

Proof. The elements of $a = \frac{1}{2}Z_3AZ_1$ can all be expressed in terms of $a_{11}(z_1, z_3) = (A_{11} + z_1A_{12} + z_3A_{21} + z_1z_3A_{22})/2$,

$$a = \begin{pmatrix} a_{11}(z_1, z_3) & a_{11}(-z_1, z_3) \\ a_{11}(z_1, -z_3) & a_{11}(-z_1, -z_3) \end{pmatrix}.$$

For A and B according to (3.4) or (3.5), the four conditions

$$|a_{11}(\pm z_1, \pm z_3)|^2 = 1$$

all reduce to the first of the relations (4.1), while for A and B according to (3.6), the first of the relations (4.2) is obtained. The remaining relations follow in a similar manner by considering b , c and d . \square

With this results, all the conditions needed to characterize the set of H_2 -reducible Hadamard matrices have been given in an explicit form. Before examining these conditions in detail, however, some additional constraints will be imposed that come from the desire to obtain a characterization in terms of inequivalent matrices.

Proposition 8. *Given A and B , the 16 possible sign combinations for the z_i parameters obtained when solving (4.1) or (4.2) generate equivalent sets of Hadamard matrices.*

Proof. The conditions (4.1) and (4.2) only determine the z_i parameters up to a sign. However, a sign change can be compensated by an interchange of rows and/or of columns, and the resulting Hadamard matrix is therefore equivalent to the original one. For instance, let H' and H'' only differ in the sign of z_3 ,

$$Z'_3 = \begin{pmatrix} 1 & z_0 \\ 1 & -z_0 \end{pmatrix} \quad \text{and} \quad Z''_3 = \begin{pmatrix} 1 & -z_0 \\ 1 & z_0 \end{pmatrix} = PZ'_3,$$

where $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a row-permuting matrix. Then

$$H'' = \begin{pmatrix} F_2 & Z_1 & Z_2 \\ Z_3'' & \frac{1}{2}Z_3''AZ_1 & \frac{1}{2}Z_3''BZ_2 \\ Z_4 & \frac{1}{2}Z_4BZ_1 & \frac{1}{2}Z_4AZ_2 \end{pmatrix} = \begin{pmatrix} e & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & e \end{pmatrix} H' \sim H'$$

□

As a consequence of Proposition 8, in order to map out the family of all non-equivalent H_2 -reducible Hadamard matrices, only one sign for the z_i parameters needs to be considered.

It can also be shown that without losing inequivalent matrices the range of the θ and ϕ parameters of Lemma 5 can be reduced to $[0, \pi)$, and the special cases corresponding to $\Lambda = \pm e$ (i.e. to Eqns (3.4) and (3.5)) can be disregarded.

Proposition 9. *Any H_2 -reducible Hadamard matrix is equivalent to a matrix on the form specified in Proposition 4, with*

$$\Lambda = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \bar{\Lambda}_{12} & -\Lambda_{11} \end{pmatrix} = \begin{pmatrix} \cos \theta & e^{i\phi} \sin \theta \\ e^{-i\phi} \sin \theta & -\cos \theta \end{pmatrix}$$

for $\theta \in [0, \pi)$, $\phi \in [0, \pi)$.

Proof. From Proposition 4 it follows that a change of sign $\Lambda \rightarrow -\Lambda$ induces the interchange $A \leftrightarrow B$,

$$H \rightarrow H' = \begin{pmatrix} F_2 & Z'_1 & Z'_2 \\ Z'_3 & \frac{1}{2}Z'_3BZ'_1 & \frac{1}{2}Z'_3AZ'_2 \\ Z'_4 & \frac{1}{2}Z'_4AZ'_1 & \frac{1}{2}Z'_4BZ'_2 \end{pmatrix}$$

When the interchange $A \leftrightarrow B$ is carried out in (4.2), the resulting equations for the z -parameters are changed. If, however, the original equations had solutions z_1, z_2, z_3 and z_4 , the new equations will have solutions $z'_1 = z_1, z'_2 = z_2, z'_3 = z_4$ and $z'_4 = z_3$. Therefore,

$$H' = \begin{pmatrix} F_2 & Z_1 & Z_2 \\ Z_4 & \frac{1}{2}Z_4BZ_1 & \frac{1}{2}Z_4AZ_2 \\ Z_3 & \frac{1}{2}Z_3AZ_1 & \frac{1}{2}Z_3BZ_2 \end{pmatrix} \sim H$$

where in the last step some rows have been permuted. Therefore, in order to map out the family of all non-equivalent H_2 -reducible Hadamard matrices, only one sign for Λ needs to be considered.

For the $\Lambda \neq \pm e$ case, the transformations $(\theta, \phi) \rightarrow (\theta + \pi, \phi)$ and $(\theta, \phi) \rightarrow (\pi - \theta, \phi + \pi)$ both imply $\Lambda \rightarrow -\Lambda$. As a result, the range for θ and ϕ can be reduced to $[0, \pi)$.

For the $\Lambda = \pm e$ case, only $\Lambda = e$ needs to be considered further, and it will first be shown that the resulting Hadamard family is equivalent to either of the two Fourier families. Indeed, from (4.1), either $z_3^2 = z_4^2 = 1$, with z_1^2 and z_2^2 unconstrained, or $z_1^2 = z_2^2 = 1$, with z_3^2 and z_4^2 unconstrained. In the first case, let $z_3 = z_4 = 1$, so that $Z_3 = Z_4 = F_2$. The resulting Hadamard matrices (see Corollary 6)

$$H = \begin{pmatrix} F_2 & Z_1 & Z_2 \\ F_2 & \omega Z_1 & \omega^2 Z_2 \\ F_2 & \omega^2 Z_1 & \omega Z_2 \end{pmatrix} \sim F_6^{(2)}$$

build the Fourier family, $F_6^{(2)}$, with z_1 and z_2 as parameters. In the second case, let $z_1 = z_2 = 1$, so that $Z_1 = Z_2 = F_2$, and the resulting matrices build the Fourier transposed family $(F_6^{(2)})^T$, with z_3 and z_4 as parameters,

$$H = \begin{pmatrix} F_2 & F_2 & F_2 \\ Z_3 & \omega Z_3 & \omega^2 Z_3 \\ Z_4 & \omega^2 Z_4 & \omega Z_4 \end{pmatrix} \sim (F_6^{(2)})^T.$$

However, as will be seen in the next section, $F_6^{(2)}$ and $(F_6^{(2)})^T$ also appear as limit families in the $\Lambda \neq \pm e$ case, for $\theta \rightarrow 0$ and $\theta \rightarrow \pi/2$. For the purpose of classifying all H_2 -reducible Hadamard matrices, the $\Lambda = \pm e$ cases can therefore be disregarded from now on. \square

5. THE THREE-PARAMETER FAMILY

Given the matrices A and B of Proposition 4, or more precisely the parameters θ and ϕ of Proposition 9, what remains is to determine in detail the conditions on the parameters z_i that follow from the unimodularity constraints (4.2). It is useful to see these constraints as Möbius transformations

$$w = \mathcal{M}(z) = \frac{\alpha z - \beta}{\beta z - \bar{\alpha}}, \quad z = \mathcal{M}^{-1}(w) = \frac{\bar{\alpha} w - \beta}{\beta w - \alpha}$$

that, as long as $|\alpha|^2 - |\beta|^2 \neq 0$, map the unit circle onto itself. Formally, from (4.2),

$$(5.1) \quad \begin{aligned} z_3^2 &= \mathcal{M}_A(z_1^2) & z_3^2 &= \mathcal{M}_B(z_2^2) \\ z_4^2 &= \mathcal{M}_A(z_2^2) & z_4^2 &= \mathcal{M}_B(z_1^2) \end{aligned}$$

with $\alpha_A = A_{12}^2$, $\beta_A = A_{11}^2$, and $\alpha_B = B_{12}^2$, $\beta_B = B_{11}^2$. Recall that the inverse of a Möbius transformation, as well as a sequence of two Möbius transformations, is also a Möbius transformation.

Through straightforward calculation, the following relation between \mathcal{M}_A and \mathcal{M}_B can easily be verified (using the expressions for A and B in terms of the Λ of Propositions 4 and 9).

Proposition 10. *For the Möbius transformations of (5.1),*

$$\mathcal{M}_B^{-1}(\mathcal{M}_A(z^2)) = \mathcal{M}_A^{-1}(\mathcal{M}_B(z^2)).$$

In view of Proposition 10, the relations (5.1) are not independent, but only allow for expressing three of the parameters z_i in terms of the fourth. Let for instance $z_1 = \exp(i\psi_1)$ where, considering Proposition 9, $\psi_1 \in [0, \pi)$. Then

$$\begin{aligned} z_3^2 &= \mathcal{M}_A(z_1^2) \\ z_4^2 &= \mathcal{M}_B(z_1^2) \\ z_2^2 &= \mathcal{M}_B^{-1}(\mathcal{M}_A(z_1^2)) = \mathcal{M}_A^{-1}(\mathcal{M}_B(z_1^2)) \end{aligned}$$

and the resulting set of Hadamard matrices will depend on the three parameters θ , ϕ and ψ_1 . The same set will be generated starting from any other z_i , and constitutes the advertised three-parameter family of complex Hadamard matrices of order 6.

The Möbius transformations (5.1) become degenerate if $|\alpha|^2 - |\beta|^2 \rightarrow 0$: the transformation $w = \mathcal{M}(z)$ degenerates into a mapping of the unit circle in z into a single point $w = \alpha/\bar{\beta}$, and this mapping has no inverse, and the inverse transform $z = \mathcal{M}^{-1}(w)$ degenerates into a mapping of the unit circle in w into a single point $z = \bar{\alpha}/\bar{\beta}$, and again there is no inverse. For \mathcal{M}_A and \mathcal{M}_A^{-1} this occurs if $|A_{11}| = |A_{12}| = 1$, i.e. if

$$\sin \theta (\sin \phi - \sqrt{3} \cos \theta \cos \phi) = 0,$$

and for \mathcal{M}_B and \mathcal{M}_B^{-1} if $|B_{11}| = |B_{12}| = 1$, i.e. if

$$\sin \theta (\sin \phi + \sqrt{3} \cos \theta \cos \phi) = 0.$$

Both transformations are degenerate when $\theta = 0$, any ϕ (and also when $\theta \rightarrow \pi$, any ϕ), and when $\theta = \pi/2$, $\phi = 0$ (and also when $\theta = \pi/2$, $\phi \rightarrow \pi$).

In general, such a degeneracy does not prevent the construction of the three-parameter family as outlined above (see Appendix 1). However, at the points where both transformations are degenerate, the analysis must take into account that these points can be reached not only along the degeneracy curves but from an arbitrary direction in the θ - ϕ plane. The resulting limit families may be obtained either through an explicit

limiting procedure, as exemplified in Appendix 2, or in the following direct manner.

If \mathcal{M}_A and \mathcal{M}_B are both degenerate, there are two cases to be considered. First, if $\theta = 0$ then $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for any ϕ , so that

$$A = F_2\Omega \quad \text{and} \quad B = F_2\Omega^2$$

Here, $\Omega = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$ with $1 + \omega + \omega^2 = 0$ and $e + \Omega + \Omega^2 = 0$ (recall that $\omega = \exp(2\pi i/3)$). The unimodularity conditions (4.2) take the form

$$(5.2) \quad \begin{aligned} (z_1^2 - \omega^4)(z_3^2 - 1) &= 0 \\ (z_2^2 - \omega^2)(z_3^2 - 1) &= 0 \\ (z_1^2 - \omega^2)(z_4^2 - 1) &= 0 \\ (z_2^2 - \omega^4)(z_4^2 - 1) &= 0 \end{aligned}$$

This set requires that $z_3^2 = 1$ and/or $z_4^2 = 1$. If $z_3^2 = z_4^2 = 1$, then there are no restrictions on z_1 or z_2 . Since all sign combinations result in equivalent Hadamard matrices, let $z_3 = z_4 = 1$. Then $Z_3 = Z_4 = F_2$, and the resulting Hadamard family is equivalent to the Fourier family $F_6^{(2)}$,

$$H = \begin{pmatrix} F_2 & Z_1 & Z_2 \\ F_2 & \Omega Z_1 & \Omega^2 Z_2 \\ F_2 & \Omega^2 Z_1 & \Omega Z_2 \end{pmatrix} \sim \begin{pmatrix} F_2 & Z_1 & Z_2 \\ F_2 & \omega Z_1 & \omega^2 Z_2 \\ F_2 & \omega^2 Z_1 & \omega Z_2 \end{pmatrix} \sim F_6^{(2)}.$$

The system (5.2) is also satisfied if $z_3^2 = 1$, $z_1^2 = \omega^2$ and $z_2^2 = \omega^4$, with z_4 arbitrary. Let $z_3 = 1$, $z_1 = \omega$ and $z_2 = \omega^2$. In this case

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \omega & -\omega & \omega^2 & -\omega^2 \\ 1 & 1 & \omega & \omega & \omega^2 & \omega^2 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & z_4 & \omega^2 & \omega^2 z_4 & \omega & \omega z_4 \\ 1 & -z_4 & \omega^2 & -\omega^2 z_4 & \omega & -\omega z_4 \end{pmatrix} \sim \begin{pmatrix} F_2 & F_2 & F_2 \\ F_2 & \omega F_2 & \omega^2 F_2 \\ Z_4 & \omega^2 Z_4 & \omega Z_4 \end{pmatrix}$$

and this family is equivalent to a subfamily of $(F_6^{(2)})^T$. Finally, the system (5.2) is also satisfied if $z_4^2 = 1$, $z_1^2 = \omega^4$ and $z_2^2 = \omega^2$, with z_3 arbitrary. Like in the previous case, the resulting H can be shown to be equivalent to a subfamily of $(F_6^{(2)})^T$.

The Möbius transformations \mathcal{M}_A and \mathcal{M}_B are also degenerate when $\theta = \pi/2$, $\phi = 0$, and in this case $\Lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and

$$A = \Omega F_2 \quad \text{and} \quad B = \Omega^2 F_2.$$

The subsequent analysis is similar to the previous one, and results in matrix families that either are equivalent to $(F_6^{(2)})^T$, or to one-parameter subfamilies of $F_6^{(2)}$.

The finding of two-parameter subfamilies at the doubly degenerate points might not have been expected, since two (θ and ϕ) of the three original parameters have been eliminated. It might be recalled, however, that a similar phenomenon was observed in [6], where the two-parameter family $K_6^{(2)}$ at certain fixed parameter values had the one-parameter $D_6^{(1)}$ family as limit family. As was detailed in [6], the extra parameter enters since the limit family depends on the direction from which the limit point is reached, just as is observed here (see Appendix 2).

It should be recalled that the appearance of the Fourier and Fourier transposed families in the present context was made use of in the proof of Proposition 9.

With these observations, the classification problem for H_2 -reducible Hadamard matrices is solved. The main results of the present paper are collected in the following theorem.

Theorem 11. *Any H_2 -reducible (complex) Hadamard matrices (of order 6) is equivalent to a member of the three-parameter family of de-phased matrices*

$$H = \begin{pmatrix} F_2 & Z_1 & Z_2 \\ Z_3 & \frac{1}{2}Z_3AZ_1 & \frac{1}{2}Z_3BZ_2 \\ Z_4 & \frac{1}{2}Z_4BZ_1 & \frac{1}{2}Z_4AZ_2 \end{pmatrix}$$

Here $F_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, A is the matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ \bar{A}_{12} & -\bar{A}_{11} \end{pmatrix}$$

with elements

$$\begin{aligned} A_{11} &= -\frac{1}{2} + i\frac{\sqrt{3}}{2}(\cos \theta + e^{-i\phi} \sin \theta) \\ A_{12} &= -\frac{1}{2} + i\frac{\sqrt{3}}{2}(-\cos \theta + e^{i\phi} \sin \theta) \end{aligned}$$

for any $\theta \in [0, \pi)$ and $\phi \in [0, \pi)$, and $B = -F_2 - A$. In the matrices $Z_i = \begin{pmatrix} 1 & 1 \\ z_i & -z_i \end{pmatrix}$, $i = 1, 2$, and $Z_i = \begin{pmatrix} 1 & z_i \\ 1 & -z_i \end{pmatrix}$, $i = 3, 4$, the parameters z_i are related through Möbius transformations

$$\begin{aligned} z_3^2 &= \mathcal{M}_A(z_1^2) & z_3^2 &= \mathcal{M}_B(z_2^2) \\ z_4^2 &= \mathcal{M}_A(z_2^2) & z_4^2 &= \mathcal{M}_B(z_1^2) \end{aligned}$$

where

$$w = \mathcal{M}(z) = \frac{\alpha z - \beta}{\bar{\beta} z - \bar{\alpha}}$$

with $\alpha_A = A_{12}^2$, $\beta_A = A_{11}^2$, and $\alpha_B = B_{12}^2$, $\beta_B = B_{11}^2$. In general, one of the parameters z_i can be chosen freely, say $z_1 = \exp(i\psi_1)$, $\psi_1 \in [0, \pi)$, in which case $z_2^2 = \mathcal{M}_A^{-1}(\mathcal{M}_B(z_1^2)) = \mathcal{M}_B^{-1}(\mathcal{M}_A(z_1^2))$, $z_3^2 = \mathcal{M}_A(z_1^2)$ and $z_4^2 = \mathcal{M}_B(z_1^2)$. Any sign combinations for z_1 , z_2 , z_3 and z_4 lead to three-parameter families that are equivalent to each other.

6. SPECIAL CASES

As pointed out above, all so far (analytically) known one- and two-parameter families of complex Hadamard matrices of order 6 are subfamilies of the three-parameter family constructed in the previous sections. In general, however, the parameters used to classify these subfamilies differ from the parameters introduced here, and the detailed connection is not always transparent. For instance, the two-parameter family $K_6^{(2)}$ of [6] exploits simplifications entailed by the assumption that $z_2 = z_1$ and $z_4 = z_3$. Such an assumption is less natural from the point of view of the parametrization developed in the present paper, and amounts to introducing a dependence between z_1 and the parameters θ and ϕ . In this respect, the family $D_6^{(1)}$ is an exception, as will be shown next.

Particularly simple subfamilies of the three-parameter family can be expected if θ and ϕ kept constant. Consider for example the point $\theta = \arccos(1/\sqrt{3})$, $\phi = \pi/4$, for which

$$\begin{cases} A_{11} &= i \\ A_{12} &= -1 \end{cases} \quad \text{and} \quad \begin{cases} B_{11} &= -1 - i \\ B_{12} &= 0 \end{cases}$$

Since $|A_{11}| = |A_{12}| = 1$, \mathcal{M}_A is degenerate and $\mathcal{M}_A(z^2) = \mathcal{M}_A^{-1}(z^2) = -1$. Furthermore, $\mathcal{M}_B(z^2) = \mathcal{M}_B^{-1}(z^2) = 1/z^2 = \bar{z}^2$ so that, taking $z_1 = z$ as independent parameter, $z_2^2 = -1$, $z_3^2 = -1$ and $z_4^2 = \bar{z}^2$. Let $z_2 = z_3 = i$ and $z_4 = \bar{z}$. The resulting one-parameter Hadamard matrix

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & z & -z & i & -i \\ 1 & i & -z & z & -1 & -i \\ 1 & -i & i & i & -i & -1 \\ 1 & \bar{z} & -i & -1 & -\bar{z} & i \\ 1 & -\bar{z} & -1 & -i & \bar{z} & i \end{pmatrix}$$

is equivalent to the generic member of $D_6^{(1)}$.

Another example of a simple subfamily can be obtained as follows. For points on the \mathcal{M}_A degeneracy curve, $\sin \phi = \sqrt{3} \cos \theta \cos \phi$ (see Section 5). Along this curve $A_{12} = \exp(2i\phi)A_{11}$ and $\mathcal{M}_A^{-1}(z^2) = \exp(-4i\phi)$, all z^2 . Let $z_1 = z = \exp(i\psi)$. If ϕ is chosen equal to $-\psi/2$ then $z_2^2 = \mathcal{M}_A^{-1}(z^2) = z^2$, i.e. $z_1 = z_2 = z$. Furthermore

$$z_3 = z_4 = \frac{1 - i\epsilon\sqrt{1+z+\bar{z}}}{1 + i\epsilon\sqrt{1+z+\bar{z}}}$$

for $\psi \in [0, 2\pi/3]$, and the resulting Hadamard matrix is equivalent to

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & z & -z & z & -z \\ 1 & z_3 & z_3 z & z & -\sqrt{z_3 z} & -\sqrt{z_3 z} \\ 1 & -z_3 & z_3 & -1 & -\sqrt{z_3 z} & \sqrt{z_3 z} \\ 1 & z_3 & -\sqrt{z_3 z} & -\sqrt{z_3 z} & z_3 z & z \\ 1 & -z_3 & -\sqrt{z_3 z} & \sqrt{z_3 z} & z_3 & -1 \end{pmatrix}$$

This one-parameter Hadamard family can be identified as a subfamily of $K_6^{(2)}$.

7. SUMMARY AND OUTLOOK

With the results of the present paper, the characterization problem for complex Hadamard matrices of order six has been given a partial solution, in that the subset of H_2 -reducible Hadamard matrices has been fully described in terms of a single, three-parameter family. There is strong numerical evidence, based on some 10^5 non-reducible Hadamard matrices (see also [2]) that a full characterization requires an additional parameter, but it remains an open question whether or not closed form expressions for such a four-parameter family can be found.

The parameters chosen here for the three-parameter family are not unique, but appear as a natural choice. Minor variations, like choosing the (SU(2)) parameters θ and ϕ differently, offer no obvious advantage.

As an application of the results presented here, Hadamard matrices in 12 dimensions can be constructed. Such an extension was outlined in [6] based on the at the time known two-parameter families in six dimensions. A corresponding extension based on the three-parameter family of the present paper results in an eleven-parameter family, the largest family constructed so far in 12 dimensions.

APPENDIX 1: DEGENERATE TRANSFORMATIONS

If one of the Möbius transformations (5.1) becomes degenerate, the three-parameter family may still be constructed as outlined in Section 5, but the result may depend on how the degeneracy limit is approached. In order to illustrate this point, let \mathcal{M}_A but not \mathcal{M}_B be degenerate. In such a case, $\mathcal{M}_A(z^2) = w_0^2$ and $\mathcal{M}_A^{-1}(z^2) = z_0^2$ for any z , where $w_0^2 = \alpha_A/\bar{\beta}_A$ and $z_0^2 = \bar{\alpha}_A/\bar{\beta}_A$ are uniquely specified by θ or ϕ along the degeneracy curve. Furthermore, $\mathcal{M}_B(z_0^2) = w_0^2$ as a consequence of Proposition 10. Using z_1 or z_4 as independent parameter, the remaining parameters are obtained through $z_4^2 = \mathcal{M}_B(z_1^2)$ with $z_3^2 = w_0^2$ and $z_2^2 = z_0^2$. On the other hand, taking z_2 or z_3 as independent parameter leads to $z_3^2 = \mathcal{M}_B(z_2^2)$ with $z_1^2 = z_0^2$ and $z_4^2 = w_0^2$. The resulting two limit matrices,

$$\begin{pmatrix} F_2 & Z_1 & Z_{z_0} \\ Z_{w_0} & \frac{1}{2}Z_{w_0}AZ_1 & \frac{1}{2}Z_{w_0}BZ_{z_0} \\ Z_4 & \frac{1}{2}Z_4BZ_1 & \frac{1}{2}Z_4AZ_{z_0} \end{pmatrix} \text{ and } \begin{pmatrix} F_2 & Z_{z_0} & Z_2 \\ Z_3 & \frac{1}{2}Z_3AZ_{z_0} & \frac{1}{2}Z_3BZ_2 \\ Z_{w_0} & \frac{1}{2}Z_{w_0}BZ_{z_0} & \frac{1}{2}Z_{w_0}AZ_2 \end{pmatrix}$$

in obvious notation, are seemingly different, but an interchange of rows and of columns shows that they generate families of equivalent Hadamard matrices. There is therefore no need to amend the general construction in Section 5 with additional rules when one of the Möbius transformations becomes degenerate.

APPENDIX 2: THE LIMIT FAMILIES AT $\theta = 0$

In order to see how the general, three-parameter family behaves when the doubly degenerate point at $\theta = 0$ is approached, let θ be infinitesimal in the expressions for A and B in Theorem 11,

$$\begin{aligned} A_{11} &\approx \omega + i\frac{\sqrt{3}}{2}e^{-i\phi}\theta & B_{11} &\approx \omega^2 - i\frac{\sqrt{3}}{2}e^{-i\phi}\theta \\ A_{12} &\approx \omega^2 + i\frac{\sqrt{3}}{2}e^{i\phi}\theta & B_{12} &\approx \omega - i\frac{\sqrt{3}}{2}e^{i\phi}\theta \end{aligned}$$

where $\omega = \exp(2\pi i/3)$. The coefficients of the Möbius transformations (5.1) are

$$\begin{aligned} \alpha_A &\approx \omega + i\sqrt{3}\omega^2 e^{i\phi\theta} & \alpha_B &\approx \omega^2 - i\sqrt{3}\omega e^{i\phi\theta} \\ \beta_A &\approx \omega^2 + i\sqrt{3}\omega e^{-i\phi\theta} & \beta_B &\approx \omega - i\sqrt{3}\omega^2 e^{-i\phi\theta} . \end{aligned}$$

Choosing z_1 as the independent parameter results in the relations, when $\theta \rightarrow 0$,

$$\begin{aligned} z_3^2 &= \mathcal{M}_A(z_1^2) \rightarrow \begin{cases} 1 & z_1^2 \neq \omega^4 \\ -1 & z_1^2 = \omega^4 \end{cases} \\ z_4^2 &= \mathcal{M}_B(z_1^2) \rightarrow \begin{cases} 1 & z_1^2 \neq \omega^2 \\ -1 & z_1^2 = \omega^2 \end{cases} \\ z_2^2 &= \mathcal{M}_A^{-1}(\mathcal{M}_B(z_1^2)) \rightarrow -e^{2i\phi} \frac{(1 + e^{-2i\phi})z_1^2 + e^{-2i\phi}}{e^{2i\phi}z_1^2 + 1 + e^{2i\phi}} \end{aligned}$$

Given z_1 (not equal to ω or ω^2), ϕ maps out a unit circle in z_2 , i.e. the Fourier family $F_6^{(2)}$ with z_1 and z_2 as independent parameters is obtained.

On the other hand, choosing z_3 as independent parameter results in

$$\begin{aligned} z_1^2 &= \mathcal{M}_A^{-1}(z_3^2) \rightarrow \begin{cases} \omega & z_3^2 \neq 1 \\ \omega^2 e^{-2i\phi} & z_3^2 = 1 \end{cases} \\ z_2^2 &= \mathcal{M}_B^{-1}(z_3^2) \rightarrow \begin{cases} \omega^2 & z_3^2 \neq 1 \\ \omega e^{-2i\phi} & z_3^2 = 1 \end{cases} \\ z_4^2 &= \mathcal{M}_A(\mathcal{M}_B^{-1}(z_3^2)) \rightarrow 1 \end{aligned}$$

Therefore, any $z_3 \neq 1$ leaves the other three parameters fixed, and, as detailed in section 6.3, the resulting Hadamard family is equivalent to a subfamily of $(F_6^{(2)})^T$.

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